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The intuitionist mathematician proposes to do mathematics as a natural function of his intellect, as a free, vital activity of thought. For him, mathematics is a production of the human mind. He uses language, both natural and formalized, only for communicating thoughts, i.e., to get others or himself to follow his own mathematical ideas. Such a linguistic accompaniment is not a representation of mathematics; still less is it mathematics itself.

It would be most in keeping with the active attitude of the intuitionist to deal at once with the construction of mathematics. The most important building block of this construction is the concept of unity which is the architectonic principle on which the series of integers depends. The integers must be treated as units which differ from one another only by their place in this series. Since in his *Logischen Grundlagen der exakten Wissenschaften* Natorp has already carried out such an analysis, which in the main conforms tolerably well to the intuitionist way of thinking, I will forego any further analysis of these concepts. But I must still make one remark which is essential for a correct understanding of our intuitionist position: we do not attribute an existence independent of our thought, i.e., a transcendental existence, to the integers or to any other mathematical objects. Even though it might be true that every thought refers to an object conceived to exist independently of it, we can nevertheless let this remain an open question. In any event, such an object need not be completely independent of human thought. Even if they should be independent of individual acts of thought, mathematical objects are by their very nature dependent on human thought. Their existence is guaranteed only insofar as they can be determined by thought. They have properties only insofar as these can be discerned in them by thought. But this possibility of knowledge is revealed to us only by the act of knowing itself. Faith in transcendental existence, unsupported by concepts, must be rejected as a means of mathematical proof. As I will shortly illustrate more

fully by an example, this is the reason for doubting the law of excluded middle.

Oskar Becker has dealt thoroughly with the problems of mathematical existence in his book on that subject. He has also uncovered many connections between these questions and the most profound philosophical problems.

We return now to the construction of mathematics. Although the introduction of the fractions as pairs of integers does not lead to any basic difficulties, the definition of the irrational numbers is another story. A real number is defined according to Dedekind by assigning to every rational number either the predicate 'Left' or the predicate 'Right' in such a way that the natural order of the rational numbers is preserved. But if we were to transfer this definition into intuitionist mathematics in exactly this form, we would have no guarantee that Euler's constant C is a real number. We do not need the definition of C. It suffices to know that this definition amounts to an algorithm which permits us to enclose C within an arbitrarily small rational interval. (A rational interval is an interval whose end points are rational numbers. But, as absolutely no ordering relations have been defined between C and the rational numbers, the word 'enclose' is obviously vague for practical purposes. The practical question is that of computing a series of rational intervals each of which is contained in the preceding one in such a way that the computation can always be continued far enough so that the last interval is smaller than an arbitrarily given limit.) But this algorithm still provides us with no way of deciding for an arbitrary rational number A whether it lies left or right of C or is perhaps equal to C. But such a method is just what Dedekind's definition, interpreted intuitionistically, would require.

The usual objection against this argument is that it does not matter whether or not this question can be decided, for, if it is not the case that $A = C$, then either $A < C$ or $A > C$, and this last alternative is decided after a finite, though perhaps unknown, number of steps N in the computation of C. I need only reformulate this objection to refute it. It can mean only this: either there exists a natural number N such that after N steps in the computation of C it turns out that $A < C$ or $A > C$; or there is no such N and hence, of course, $A = C$. But, as we have seen, the existence of N signifies nothing but the possibility of actually producing a number with the requisite property, and the non-existence of N signifies the possibility of deriving a contradiction from this property. Since we do not know whether or not one of these possibilities exists, we may not assert that N either exists or does not exist. In this sense, we can say that the law of excluded middle may not be used here.

In its original form, then, Dedekind's definition cannot be used in intuitionist mathematics. Brouwer, however, has improved it in the following way: Think of the rational numbers enumerated in some way. For the sake of simplicity, we restrict ourselves to the numbers in the closed unit interval and take always as our basis the following enumeration:

$$\text{(A)}\quad 0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

A real number is determined by a cut in the series (A); i.e., by a rule which assigns to each rational number in the series either the predicate 'Left' or the predicate 'Right' in such a way that the natural order of the rational numbers is preserved. At each step, however, we permit one individual number to be left out of this mapping. For example, let the rule be so formed that the series of predicates begins this way:

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots$$

L, R, L, L, ?, L,

Here $\frac{2}{3}$ is temporarily left out of the mapping. We need not know whether or not the predicate for $\frac{2}{3}$ is ever determined. But it is also a possibility that $\frac{3}{4}$ should become a new excluded number and hence that $\frac{2}{3}$ would receive the predicate 'Left'.

It is easy to give a cut for Euler's constant. Let d_n be the smallest difference between two successive numbers in the first n numbers of (A). Now if we compute C far enough to get a rational interval i which is smaller than d_n, then at most one of these n numbers can fall within i. If there is such a number, it becomes the excluded number for the cut. Thus, we can see how closely Brouwer's definition is related to the actual computation of a real number.

We can now take an important step forward. We can drop the requirement that the series of predicates be determined to infinity by a rule. It suffices if the series is determined step by step in some way, e.g., by free choices. I call such sequences "infinitely proceeding." Thus the definition of real numbers is extended to allow infinitely proceeding sequences in addition to rule-determined sequences. Before discussing this new definition in detail, we will give a simple example.We begin with this"Left-Right"choice-sequence:

$$0, 1, \frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5},$$

L, R, L, L, R, L, R, L, L,

Here the question about which predicate $\frac{3}{5}$ receives cannot be answered yet, for it must still be decided which predicate to give it. The question about the predicate which $\frac{3}{4}$ receives, on the other hand, can be answered now by 'Right', since that choice would hold for every possible continuation of the sequence. In general, only those questions about an infinitely proceeding

sequence which refer to every possible continuation of the sequence are susceptible of a determinate answer. Other questions, like the foregoing about the predicate for $\frac{3}{5}$, must therefore be regarded as meaningless. Thus choice-sequences supplant, not so much the individual rule-determined sequences, but rather the totality of all possible rules. A "Left-Right" choice-sequence, the freedom of choice for which is limited only by the conditions which result from the natural order of the rational numbers, determines not just one real number but the spread of all real numbers or the continuum. Whereas we ordinarily think of each real number as individually defined and only afterwards think of them all together, we here define the continuum as a totality. If we restrict this freedom of choice by rules given in advance, we obtain spreads of real numbers. For example, if we prescribe that the sequence begin in the way we have just written it, we define the spread of real numbers between $\frac{1}{2}$ and $\frac{2}{3}$. An infinitely proceeding sequence gradually becomes a rule-determined sequence when more and more restrictions are placed on the freedom of choice.

We have used the word 'spread' exactly in Brouwer's sense. His definition of a spread is a generalization of this notion. In addition to choice-sequences, Brouwer treats sequences which are formed from choice-sequences by mapping-rules. A spread involves two rules. The first rule states which choices of natural numbers are allowed after a determinate finite series of permitted choices has been made. The rule must be so drawn that at least one new permissible choice is known after each finite series of permitted choices has been made. The natural order of the rational numbers is an example of such a rule for our "Left-Right" sequence previously given. The second rule involved in a spread assigns a mathematical object to each permissible choice. The mathematical object may, of course, depend also on choices previously made. Thus it is permissible to terminate the mapping at some particular number and to assign nothing to subsequent choices. A sequence which results from a permissible choice-sequence by a mapping-rule is called an "element" of the spread.

To bring our previous example of the spread of real numbers between $\frac{1}{2}$ and $\frac{2}{3}$ under this general definition, we will replace the predicates 'Left', 'Right', and 'temporarily undetermined', by 1, 2, and 3; and we will derive the rule for permissible choices from the natural order of the rational numbers and from the requirement that the sequence begin in a particular way; and we will take identity for the mapping-rule.

A spread is not the sum of its elements (this statement is meaningless unless spreads are regarded as existing in themselves). Rather, a spread is identified with its defining rules. Two elements of a spread are said to be equal if equal objects exist at the nth place in both for every n. Equality of elements of a spread, therefore, does not mean that they are the same element. To be the same, they would have to be assigned to the same spread by the same choice-sequence. It would be impractical to call two mathematical objects equal only if they are the same object. Rather, every kind of object must receive its own definition of equality.

Brouwer calls "species" those spreads which are defined, in classical terminology, by a characteristic property of their members. A species, like a spread, is not regarded as the sum of its members but is rather identified with its defining property. Impredicative definitions are made impossible by the fact, which intuitionists consider self-evident, that only previously defined objects may occur as members of a species. There results, consequently, a step-by-step introduction of species. The first level is made up of those spread-species whose defining property is identity with an element of a particular spread. Hence, to every spread M there corresponds the spread-species of those spread-elements which are identical with some element of M.[1] A species of the first order can contain spread-elements and spread-species. In addition, a species of the second order contains species of the first order as members, and so on.

The introduction of infinitely proceeding sequences is not a necessary consequence of the intuitionist approach. Intuitionist mathematics could be constructed without choice-sequences. But the following set-theoretic theorem about the continuum shows how much mathematics would thereby be impoverished. This theorem will also serve as an example of an intuitionist reasoning process.

Let there be a rule assigning to each real number a natural number as its correlate. Assume that the real numbers a and b have different correlates, e.g., 1 and 2. Then, by a simple construction, we can determine a third number c which has the following property: in every neighborhood of c, no matter how small, there is a mapped number other than c; i.e., every finite initial segment of the cut which defines c can be continued so as to get a mapped number other than c. We define the number d by a choice-sequence thus: we begin as with c but we reserve the freedom to continue at an arbitrary choice in a way different from that for c. Obviously the correlate of d is not determined after any previously known finite number of choices. Accordingly, no definite correlate is assigned to d. But this conclusion contradicts our premise that every real number has a correlate. Our assumption that the two numbers a and b have different correlates is thus shown to be contradictory. And, since two natural numbers which cannot be distinguished are the same number, we have the following theorem: if every real number is assigned a correlate, then all the real numbers have the same correlate.

As a special result, we have: if a continuum is divided into two subspecies in such a way that every member belongs to one and only one of these subspecies, then one of the subspecies is empty and the other is identical with the continuum.

The unit continuum, for example, cannot be subdivided into the species of numbers between 0 and $\frac{1}{2}$ and the species of numbers between $\frac{1}{2}$ and 1,

[1] This definition of spread-species is taken from a communication of Professor Brouwer.

for the preceding construction produces a number for which one need never decide whether it is larger or smaller than $\frac{1}{2}$. The theorems about the continuity of a function determined in an interval are also connected with the foregoing theorem. But Brouwer's theorem about the uniform continuity of all full functions goes far beyond these results.

But what becomes of the theorem we have just proved if no infinitely proceeding sequences are allowed in mathematics? In that event, the species of numbers defined by rule-determined sequences would have to take the place of the continuum. This definition is admissible if we take it to mean that a number belongs to this species only if there is a rule which permits us actually to determine all the predicates of the sequence successively.

In this event, the foregoing proof continues to hold only if we succeed in defining the number d by a rule-determined sequence rather than by a choice-sequence. We can probably do it if we make use of certain unsolved problems; e.g., whether or not the sequence 0123456789 occurs in the decimal expansion of π. We can let the question—whether or not to deviate from the predicate series for c, at the nth predicate in the predicate sequence for d—depend on the occurrence of the preceding sequence at the nth digit after the decimal point in π. This proof obviously is weakened as soon as the question about the sequence is answered. But, in the event that it is answered, we can replace this question by a similar unanswered question, if there are any left. We can prove our theorem for rule-determined sequences only on the condition that there always remain unsolved problems. More precisely, the theorem is true if there are two numbers, determined by rule-determined sequences, such that the question about whether they are the same or different poses a demonstrably unsolvable problem. It is false if the existence of two such numbers is contradictory. But the problem raised by these conditions is insuperable. Even here choice-sequences prove to be superior to rule-determined sequences in that the former make mathematics independent of the question of the existence of unsolvable problems.

We conclude our treatment of the construction of mathematics in order to say something about the intuitionist propositional calculus. We here distinguish between propositions and assertions. An assertion is the affirmation of a proposition. A mathematical proposition expresses a certain expectation. For example, the proposition, 'Euler's constant C is rational', expresses the expectation that we could find two integers a and b such that $C = a/b$. Perhaps the word 'intention', coined by the phenomenologists, expresses even better what is meant here. We also use the word 'proposition' for the intention which is linguistically expressed by the proposition. The intention, as already emphasized above, refers not only to a state of affairs thought to exist independently of us but also to an experience thought to be possible, as the preceding example clearly brings out.

The affirmation of a proposition means the fulfillment of an intention. The assertion 'C is rational', for example, would mean that one has in fact found the desired integers. We distinguish an assertion from its corresponding proposition by the assertion sign '$\vdash$' that Frege introduced and which Russell and Whitehead also used for this purpose. The affirmation of a proposition is not itself a proposition; it is the determination of an empirical fact, viz., the fulfillment of the intention expressed by the proposition.

A logical function is a process for forming another proposition from a given proposition. Negation is such a function. Becker, following Husserl, has described its meaning very clearly. For him negation is something thoroughly positive, viz., the intention of a contradiction contained in the original intention. The proposition 'C is not rational', therefore, signifies the expectation that one can derive a contradiction from the assumption that C is rational. It is important to note that the negation of a proposition always refers to a proof procedure which leads to the contradiction, even if the original proposition mentions no proof procedure. We use $\neg$ as the symbol for negation.

For the law of excluded middle we need the logical function "either-or". '$p \vee q$' signifies that intention which is fulfilled if and only if at least one of the intentions p and q is fulfilled. The formula for the law of excluded middle would be '$\vdash p \vee \neg p$'. One can assert this law for a particular proposition p only if p either has been proved or reduced to a contradiction. Thus, a proof that the law of excluded middle is a general law must consist in giving a method by which, when given an arbitrary proposition, one could always prove either the proposition itself or its negation. Thus the formula '$p \vee \neg p$' signifies the expectation of a mathematical construction (method of proof) which satisfies the aforementioned requirement. Or, in other words, this formula is a mathematical proposition; the question of its validity is a mathematical problem which, when the law is stated generally, is unsolvable by mathematical means. In this sense, logic is dependent on mathematics.

We conclude with some remarks on the question of the solvability of mathematical problems. A problem is posed by an intention whose fulfillment is sought. It is solved either if the intention is fulfilled by a construction or if it is proved that the intention leads to a contradiction. The question of solvability can, therefore, be reduced to that of provability.

A proof of a proposition is a mathematical construction which can itself be treated mathematically. The intention of such a proof thus yields a new proposition. If we symbolize the proposition 'the proposition p is provable' by '$+p$', then '$+$' is a logical function, viz., "provability". The assertions '$\vdash p$' and '$\vdash +p$' have exactly the same meaning. For, if p is proved, the provability of p is also proved, and if $+p$ is proved, then the intention of a proof of p has been fulfilled, i.e., p has been proved. Nevertheless, the propositions p and $+p$ are not identical, as can best be made clear by an example. In the computation of Euler's constant C, it can happen that a particular rational value, say A, is contained for an unusually long time within the interval within which we keep more narrowly enclosing C so that

we finally suspect that $C = A$; i.e., we expect that, if we continued the computation of C, we would keep on finding A within this interval. But such a suspicion is by no means a proof that it will always happen. The proposition $+(C = A)$, therefore, contains more than the proposition $(C = A)$.

If we apply negation to both of these propositions, then we get not only two different propositions, '$\neg p$' and '$\neg +p$', but also the assertions, '$\vdash \neg p$' and '$\vdash \neg +p$', are different. '$\vdash \neg +p$' means that the assumption of such a construction as $+p$ requires is contradictory. The simple expectation p, however, need not lead to a contradiction. Here is how this works in our example just cited. Assume that we have proved the contradictoriness of the assumption that there is a construction which proves that A lies within every interval that contains C ($\vdash \neg +p$). But still the assumption that in the actual computation of C we will always in fact find A within our interval need not lead to a contradiction. It is even conceivable that we might prove that the latter assumption could never be proved to be contradictory, and hence that we could assert at the same time both '$\vdash \neg +p$' and '$\vdash \neg \neg p$'. In such an event, the problem whether $C = A$ would be essentially unsolvable.

The distinction between p and $+p$ vanishes as soon as a construction is intended in p itself, for the possibility of a construction can be proved only by its actual execution. If we limit ourselves to those propositions which require a construction, the logical function of provability generally does not arise. We can impose this restriction by treating only propositions of the form 'p is provable' or, to put it another way, by regarding every intention as having the intention of a construction for its fulfillment added to it. It is in this sense that intuitionist logic, insofar as it has been developed up to now without using the function $+$, must be understood. The introduction of provability would lead to serious complications. Yet its minimal practical value would hardly make it worthwhile to deal with those complications in detail.[2] But here this notion has given us an insight into how to conceive of essentially unsolvable problems.

We will have accomplished our purpose if we have shown you that intuitionism contains no arbitrary assumptions. Still less does it contain artificial prohibitions, such as those used to avoid the logical paradoxes. Rather, once its basic attitude has been adopted, intuitionism is the only possible way to construct mathematics.

[2] The question dealt with in this paragraph was fully clarified only in a discussion with H. Freudenthal after the conference. The results of this discussion are reproduced in the text.