a system of things: to know that this system includes the harmony of logical rationality, and the harmony of aesthetic achievement: to know that, while the harmony of logic lies upon the universe as an iron necessity, the aesthetic harmony stands before it as a living ideal moulding the general flux in its broken progress towards finer, subtler issues.

CHAPTER II

MATHEMATICS AS AN ELEMENT IN THE HISTORY OF THOUGHT

THE science of Pure Mathematics, in its modern developments, may claim to be the most original creation of the human spirit. Another claimant for this position is music. But we will put aside all rivals, and consider the ground on which such a claim can be made for mathematics. The originality of mathematics consists in the fact that in mathematical science connections between things are exhibited which, apart from the agency of human reason, are extremely unobvious. Thus the ideas, now in the minds of contemporary mathematicians, lie very remote from any notions which can be immediately derived by perception through the senses; unless indeed it be perception stimulated and guided by antecedent mathematical knowledge. This is the thesis which I proceed to exemplify.

Suppose we project our imagination backwards through many thousands of years, and endeavour to realise the simple-mindedness of even the greatest intellects in those early societies. Abstract ideas which to us are immediately obvious must have been, for them, matters only of the most dim apprehension. For example take the question of number. We think of the number 'five' as applying to appropriate groups of any entities whatsoever—to five fishes, five children, five apples, five days. Thus in considering the relations of the number 'five' to the number 'three,' we are thinking of two groups of things, one with five members and the other with three members. But we are entirely abstracting from any consideration of any particular entities, or even of any particular sorts of entities, which go to make up the membership of either of the two groups. We are merely thinking of those relationships between those two groups which are entirely independent of the individual essences of any of the members of either group. This is a very remarkable feat of abstraction; and it must have taken ages for the human race to rise to it. During a long period, groups of fishes will have been compared to each other in respect to their multiplicity, and groups of days to each other. But the first man who noticed the analogy between a group of seven fishes and a group of seven days made a notable advance in the history of thought. He was the first man who entertained a concept belonging to the science of pure mathematics. At that moment it must have been impossible for him to divine the complexity and subtlety of these abstract mathematical ideas which were waiting for discovery. Nor could he have guessed that these notions would exert a widespread fascination in each succeeding generation. There is an erroneous literary tradition which represents the love of mathematics as a monomania confined to a few eccentrics in each generation. But be this as it may, it would have been impossible to anticipate the pleasure derivable from a type of abstract thinking which had no counterpart in the then-existing society. Thirdly, the tremendous future effect of mathematical knowledge on the lives of men, on their daily avocations, on their habitual thoughts, on the organisation of society, must have been even more completely shrouded from the foresight of those early thinkers. Even now there is a very wavering grasp of the true position of mathematics as an element in the history of thought. I will not go so far as to say that to construct a history of thought without profound study of the mathematical ideas of successive epochs is like omitting

Hamlet from the play which is named after him. That would be claiming too much. But it is certainly analogous to cutting out the part of Ophelia. This simile is singularly exact. For Ophelia is quite essential to the play, she is very charming,—and a little mad. Let us grant that the pursuit of mathematics is a divine madness of the human spirit, a refuge from the goading urgency of contingent happenings.

When we think of mathematics, we have in our mind a science devoted to the exploration of number, quantity, geometry, and in modern times also including investigation into yet more abstract concepts of order, and into analogous types of purely logical relations. The point of mathematics is that in it we have always got rid of the particular instance, and even of any particular sorts of entities. So that for example, no mathematical truths apply merely to fish, or merely to stones, or merely to colours. So long as you are dealing with pure mathematics, you are in the realm of complete and absolute abstraction. All you assert is, that reason insists on the admission that, if any entities whatever have any relations which satisfy such-and-such purely abstract conditions, then they must have other relations which satisfy other purely abstract conditions.

Mathematics is thought moving in the sphere of complete abstraction from any particular instance of what it is talking about. So far is this view of mathematics from being obvious, that we can easily assure ourselves that it is not, even now, generally understood. For example, it is habitually thought that the certainty of mathematics is a reason for the certainty of our geometrical knowledge of the space of the physical universe. This is a delusion which has vitiated much philosophy in the past, and some philosophy in the present. This question of geometry is a test case of some urgency. There are certain alternative sets of purely abstract conditions possible for the relationship of groups of unspecified entities, which I will call *geometrical conditions*. I give them this name because of their general analogy to those conditions, which we believe to hold respecting the particular geometrical relations of things observed by us in our direct perception of nature. So far as our observations are concerned, we are not quite accurate enough to be certain of the exact conditions regulating the things we come across in nature. But we can by a slight stretch of hypothesis identify these observed conditions with some one set of the purely abstract geometrical conditions. In doing so, we make a particular determination of the group of unspecified entities which are the *relata* in the abstract science. In the pure mathematics of geometrical relationships, we say that, if *any* group entities enjoy *any* relationships among its members satisfying *this* set of abstract geometrical conditions, then such-and-such additional abstract conditions must also hold for such relationships. But when we come to physical space, we say that some definitely observed group of physical entities enjoys some definitely observed relationships among its members which do satisfy this above-mentioned set of abstract geometrical conditions. We thence conclude that the additional relationships which we concluded to hold in *any* such case, must therefore hold in *this particular* case.

The certainty of mathematics depends upon its complete abstract generality. But we can have no *à priori* certainty that we are right in believing that the observed entities in the concrete universe form a particular instance of what falls under our general reasoning. To take another example from arithmetic. It is a general abstract truth of pure mathematics that any group of forty entities can be subdivided into two groups of twenty entities. We are therefore justified in concluding that a particular group of apples which

we believe to contain forty members can be subdivided into two groups of apples of which each contains twenty members. But there always remains the possibility that we have miscounted the big group; so that, when we come in practice to subdivide it, we shall find that one of the two heaps has an apple too few or an apple too many.

Accordingly, in criticising an argument based upon the application of mathematics to particular matters of fact there are always three processes to be kept perfectly distinct in our minds. We must first scan the purely mathematical reasoning to make sure that there are no mere slips in it—no casual illogicalities due to mental failure. Any mathematician knows from bitter experience that, in first elaborating a train of reasoning, it is very easy to commit a slight error which yet makes all the difference. But when a piece of mathematics has been revised, and has been before the expert world for some time, the chance of a casual error is almost negligible. The next process is to make quite certain of all the abstract conditions which have been presupposed to hold. This is the determination of the abstract premises from which the mathematical reasoning proceeds. This is a matter of considerable difficulty. In the past quite remarkable oversights have been made, and have been accepted by generations of the greatest mathematicians. The chief danger is that of oversight, namely, tacitly to introduce some condition, which it is natural for us to presuppose, but which in fact need not always be holding. There is another opposite oversight in this connection which does not lead to error, but only to lack of simplification. It is very easy to think that more postulated conditions are required than is in fact the case. In other words, we may think that some abstract postulate is necessary which is in fact capable of being proved from the other postulates that we have already on hand. The only effects of this excess of abstract postulates are to diminish our aesthetic pleasure in the mathematical reasoning, and to give us more trouble when we come to the third process of criticism.

This third process of criticism is that of verifying that our abstract postulates hold for the particular case in question. It is in respect to this process of verification for the particular case that all the trouble arises. In some simple instances, such as the counting of forty apples, we can with a little care arrive at practical certainty. But in general, with more complex instances, complete certainty is unattainable. Volumes, libraries of volumes, have been written on the subject. It is the battle-ground of rival philosophers. There are two distinct questions involved. There are particular definite things observed, and we have to make sure that the relations between these things really do obey certain definite exact abstract conditions. There is great room for error here. The exact observational methods of science are all contrivances for limiting these erroneous conclusions as to direct matters of fact. But another question arises. The things directly observed are, almost always, only samples. We want to conclude that the abstract conditions, which hold for the samples, also hold for all other entities which, for some reason or other, appear to us to be of the same sort. This process of reasoning from the sample to the whole species is Induction. The theory of Induction is the despair of philosophy—and yet all our activities are based upon it. Anyhow, in criticising a mathematical conclusion as to a particular matter of fact, the real difficulties consist in finding out the abstract assumptions involved, and in estimating the evidence for their applicability to the particular case in hand.

It often happens, therefore, that in criticising a learned book of applied mathematics, or a memoir, one's whole

trouble is with the first chapter, or even with the first page. For it is there, at the very outset, where the author will probably be found to slip in his assumptions. Further, the trouble is not with what the author does say, but with what he does not say. Also it is not with what he knows he has assumed, but with what he has unconsciously assumed. We do not doubt the author's honesty. It is his perspicacity which we are criticising. Each generation criticises the unconscious assumptions made by its parents. It may assent to them, but it brings them out in the open.

The history of the development of language illustrates this point. It is a history of the progressive analysis of ideas. Latin and Greek were inflected languages. This means that they express an unanalysed complex of ideas by the mere modification of a word; whereas in English, for example, we use prepositions and auxiliary verbs to drag into the open the whole bundle of ideas involved. For certain forms of literary art,—though not always—the compact absorption of auxiliary ideas into the main word may be an advantage. But in a language such as English there is the overwhelming gain in explicitness. This increased explicitness is a more complete exhibition of the various abstractions involved in the complex idea which is the meaning of the sentence.

By comparison with language, we can now see what is the function in thought which is performed by pure mathematics. It is a resolute attempt to go the whole way in the direction of complete analysis, so as to separate the elements of mere matter of fact from the purely abstract conditions which they exemplify.

The habit of such analysis enlightens every act of the functioning of the human mind. It first (by isolating it) emphasizes the direct aesthetic appreciation of the content of experience. This direct appreciation means an apprehension of what this experience is in itself in its own particular essence, including its immediate concrete values. This is a question of direct experience, dependent upon sensitive subtlety. There is then the abstraction of the particular entities involved, viewed in themselves, and as apart from that particular occasion of experience in which we are then apprehending them. Lastly there is the further apprehension of the absolutely general conditions satisfied by the particular relations of those entities as in that experience. These conditions gain their generality from the fact that they are expressible without reference to those particular relations or to those particular *relata* which occur in that particular occasion of experience. They are conditions which might hold for an indefinite variety of other occasions, involving other entities and other relations between them. Thus these conditions are perfectly general because they refer to no particular occasion, and to no particular entities (such as green, or blue, or trees) which enter into a variety of occasions, and to no particular relationships between such entities.

There is, however, a limitation to be made to the generality of mathematics; it is a qualification which applies equally to all general statements. No statement, except one, can be made respecting any remote occasion which enters into no relationship with the immediate occasion so as to form a constitutive element of the essence of that immediate occasion. By the 'immediate occasion' I mean that occasion which involves as an ingredient the individual act of judgment in question. The one excepted statement is,—If anything out of relationship, then complete ignorance as to it. Here by 'ignorance,' I mean *ignorance*; accordingly no advice can be given as to how to expect it, or to treat it,

in 'practice' or in any other way. Either we know something of the remote occasion by the cognition which is itself an element of the immediate occasion, or we know nothing. Accordingly the full universe, disclosed for every variety of experience, is a universe in which every detail enters into its proper relationship with the immediate occasion. The generality of mathematics is the most complete generality consistent with the community of occasions which constitutes our metaphysical situation.

It is further to be noticed that the particular entities require these general conditions for their ingression into any occasions; but the same general conditions may be required by many types of particular entities. This fact, that the general conditions transcend any one set of particular entities, is the ground for the entry into mathematics, and into mathematical logic, of the notion of the 'variable.' It is by the employment of this notion that general conditions are investigated without any specification of particular entities. This irrelevance of the particular entities has not been generally understood: for example, the shape-iness of shapes, *e.g.* circularity and sphericity and cubicality as in actual experience, do not enter into the geometrical reasoning.

The exercise of logical reason is always concerned with these absolutely general conditions. In its broadest sense, the discovery of mathematics is the discovery that the totality of these general abstract conditions, which are concurrently applicable to the relationships among the entities of any one concrete occasion, are themselves inter-connected in the manner of a pattern with a key to it. This pattern of relationships among general abstract conditions is imposed alike on external reality, and on our abstract representations of it, by the general necessity that every thing must be just its own individual self, with its own individual way of differing from everything else. This is nothing else than the necessity of abstract logic, which is the presupposition involved in the very fact of inter-related existence as disclosed in each immediate occasion of experience.

The key to the patterns means this fact:—that from a select set of those general conditions, exemplified in any one and the same occasion, a pattern involving an infinite variety of other such conditions, also exemplified in the same occasion, can be developed by the pure exercise of abstract logic. Any such select set is called the set of postulates, or premises, from which the reasoning proceeds. The reasoning is nothing else than the exhibition of the whole pattern of general conditions involved in the pattern derived from the selected postulates.

The harmony of the logical reason, which divines the complete pattern as involved in the postulates, is the most general aesthetic property arising from the mere fact of concurrent existence in the unity of one occasion. Wherever there is a unity of occasion there is thereby established an aesthetic relationship between the general conditions involved in that occasion. This aesthetic relationship is that which is divined in the exercise of rationality. Whatever falls within that relationship is thereby exemplified in that occasion, whatever falls without that relationship is thereby excluded from exemplification in that occasion. The complete pattern of general conditions, thus exemplified, is determined by any one of many select sets of these conditions. These key sets are sets of equivalent postulates. This reasonable harmony of being, which is required for the unity of a complex occasion, together with the completeness of the realisation (in that occasion) of all that is involved in its logical harmony, is the primary article of metaphysical doctrine. It means that for things to be together

involves that they are reasonably together. This means that thought can penetrate into every occasion of fact, so that by comprehending its key conditions, the whole complex of its pattern of conditions lies open before it. It comes to this:—provided we know something which is perfectly general about the elements in any occasion, we can then know an indefinite number of other equally general concepts which must also be exemplified in that same occasion. The logical harmony involved in the unity of an occasion is both exclusive and inclusive. The occasion must exclude the inharmonious, and it must include the harmonious.

Pythagoras was the first man who had any grasp of the full sweep of this general principle. He lived in the sixth century before Christ. Our knowledge of him is fragmentary. But we know some points which establish his greatness in the history of thought. He insisted on the importance of the utmost generality in reasoning, and he divined the importance of number as an aid to the construction of any representation of the conditions involved in the order of nature. We know also that he studied geometry, and discovered the general proof of the remarkable theorem about right-angled triangles. The formation of the Pythagorean Brotherhood, and the mysterious rumours as to its rites and its influence, afford some evidence that Pythagoras divined, however dimly, the possible importance of mathematics in the formation of science. On the side of philosophy he started a discussion which has agitated thinkers ever since. He asked, "What is the status of mathematical entities, such as numbers for example, in the realm of things?" The number 'two,' for example, is in some sense exempt from the flux of time and the necessity of position in space. Yet it is involved in the real world. The same considerations apply to geometrical notions—to circular shape, for example.

Pythagoras is said to have taught that the mathematical entities, such as numbers and shapes, were the ultimate stuff out of which the real entities of our perceptual experience are constructed. As thus baldly stated, the idea seems crude, and indeed silly. But undoubtedly, he had hit upon a philosophical notion of considerable importance; a notion which has a long history, and which has moved the minds of men, and has even entered into Christian theology. About a thousand years separate the Athanasian Creed from Pythagoras, and about two thousand four hundred years separate Pythagoras from Hegel. Yet for all these distances in time, the importance of definite number in the constitution of the Divine Nature, and the concept of the real world as exhibiting the evolution of an idea, can both be traced back to the train of thought set going by Pythagoras.

The importance of an individual thinker owes something to chance. For it depends upon the fate of his ideas in the minds of his successors. In this respect Pythagoras was fortunate. His philosophical speculations reach us through the mind of Plato. The Platonic world of ideas is the refined, revised form of the Pythagorean doctrine that number lies at the base of the real world. Owing to the Greek mode of representing numbers by patterns of dots, the notions of number and of geometrical configuration are less separated than with us. Also Pythagoras, without doubt, included the shape-iness of shape, which is an impure mathematical entity. So to-day, when Einstein and his followers proclaim that physical facts, such as gravitation, are to be construed as exhibitions of local peculiarities of spatio-temporal properties, they are following the pure Pythagorean tradition. In a sense, Plato and Pythagoras stand nearer to modern physical science than does Aristotle. The two former were mathematicians, whereas Aristotle was the son of a doctor,

though of course he was not thereby ignorant of mathematics. The practical counsel to be derived from Pythagoras, is to measure, and thus to express quality in terms of numerically determined quantity. But the biological sciences, then and till our own time, have been overwhelmingly classificatory. Accordingly, Aristotle by his Logic throws the emphasis on classification. The popularity of Aristotelian Logic retarded the advance of physical science throughout the Middle Ages. If only the schoolmen had measured instead of classifying, how much they might have learnt!

Classification is a half-way house between the immediate concreteness of the individual thing and the complete abstraction of mathematical notions. The species take account of the specific character, and the genera of the generic character. But in the procedure of relating mathematical notions to the facts of nature, by counting, by measurement, and by geometrical relations, and by types of order, the rational contemplation is lifted from the incomplete abstractions involved in definite species and genera, to the complete abstractions of mathematics. Classification is necessary. But unless you can progress from classification to mathematics, your reasoning will not take you very far.

Between the epoch which stretches from Pythagoras to Plato and the epoch comprised in the seventeenth century of the modern world nearly two thousand years elapsed. In this long interval mathematics had made immense strides. Geometry had gained the study of conic sections and trigonometry; the method of exhaustion had almost anticipated the integral calculus; and above all the Arabic arithmetical notation and algebra had been contributed by Asiatic thought. But the progress was on technical lines. Mathematics, as a formative element in the development of philosophy, never, during this long period, recovered from its deposition at the hands of Aristotle. Some of the old ideas derived from the Pythagorean-Platonic epoch lingered on, and can be traced among the Platonic influences which shaped the first period of evolution of Christian theology. But philosophy received no fresh inspiration from the steady advance of mathematical science. In the seventeenth century the influence of Aristotle was at its lowest, and mathematics recovered the importance of its earlier period. It was an age of great physicists and great philosophers; and the physicists and philosophers were alike mathematicians. The exception of John Locke should be made; although he was greatly influenced by the Newtonian circle of the Royal Society. In the age of Galileo, Descartes, Spinoza, Newton, and Leibniz, mathematics was an influence of the first magnitude in the formation of philosophic ideas. But the mathematics, which now emerged into prominence, was a very different science from the mathematics of the earlier epoch. It had gained in generality, and had started upon its almost incredible modern career of piling subtlety of generalisation upon subtlety of generalisation; and of finding, with each growth of complexity, some new application, either to physical science, or to philosophic thought. The Arabic notation had equipped the science with almost perfect technical efficiency in the manipulation of numbers. This relief from a struggle with arithmetical details (as instanced, for example, in the Egyptian arithmetic of B.C. 1600) gave room for a development which had already been faintly anticipated in later Greek mathematics. Algebra now came upon the scene, and algebra is a generalisation of arithmetic. In the same way as the notion of number abstracted from reference to any one particular set of entities, so in algebra abstraction is made from the notion of any particular numbers. Just as the number '5' refers impartially to any

group of five entities, so in algebra the letters are used to refer impartially to any number, with the proviso that each letter is to refer to the same number throughout the same context of its employment.

This usage was first employed in equations, which are methods of asking complicated arithmetical questions. In this connection, the letters representing numbers were termed 'unknowns.' But equations soon suggested a new idea, that, namely, of a function of one or more general symbols, these symbols being letters representing any numbers. In this employment the algebraic letters are called the 'arguments' of the function, or sometimes they are called the 'variables.' Then, for instance, if an angle is represented by an algebraical letter, as standing for its numerical measure in terms of a given unit, Trigonometry is absorbed into this new algebra. Algebra thus develops into the general science of analysis in which we consider the properties of various functions of undetermined arguments. Finally the particular functions, such as the trigonometrical functions, and the logarithmic functions, and the algebraic functions, are generalised into the idea of 'any function.' Too large a generalisation leads to mere barrenness. It is the large generalisation, limited by a happy particularity, which is the fruitful conception. For instance the idea of any *continuous* function, whereby the limitation of continuity is introduced, is the fruitful idea which has led to most of the important applications. This rise of algebraic analysis was concurrent with Descartes' discovery of analytical geometry, and then with the invention of the infinitesimal calculus by Newton and Leibniz. Truly, Pythagoras, if he could have foreseen the issue of the train of thought which he had set going would have felt himself fully justified in his brotherhood with its excitement of mysterious rites.

The point which I now want to make is that this dominance of the idea of functionality in the abstract sphere of mathematics found itself reflected in the order of nature under the guise of mathematically expressed laws of nature. Apart from this progress of mathematics, the seventeenth century developments of science would have been impossible. Mathematics supplied the background of imaginative thought with which the men of science approached the observation of nature. Galileo produced formulae, Descartes produced formulae, Huyghens produced formulae, Newton produced formulae.

As a particular example of the effect of the abstract development of mathematics upon the science of those times, consider the notion of periodicity. The general recurrences of things are very obvious in our ordinary experience. Days recur, lunar phases recur, the seasons of the year recur, rotating bodies recur to their old positions, beats of the heart recur, breathing recurs. On every side, we are met by recurrence. Apart from recurrence, knowledge would be impossible; for nothing could be referred to our past experience. Also, apart from some regularity of recurrence, measurement would be impossible. In our experience, as we gain the idea of exactness, recurrence is fundamental.

In the sixteenth and seventeenth centuries, the theory of periodicity took a fundamental place in science. Kepler divined a law connecting the major axes of the planetary orbits with the periods in which the planets respectively described their orbits: Galileo observed the periodic vibrations of pendulums: Newton explained sound as being due to the disturbance of air by the passage through it of periodic waves of condensation and rarefaction: Huyghens explained light as being due to the transverse waves of vibration of a subtle ether; Mersenne connected the period

of the vibration of a violin string with its density, tension, and length. The birth of modern physics depended upon the application of the abstract idea of periodicity to a variety of concrete instances. But this would have been impossible, unless mathematicians had already worked out in the abstract the various abstract ideas which cluster round the notions of periodicity. The science of trigonometry arose from that of the relations of the angles of a right-angled triangle, to the ratios between the sides and hypotenuse of the triangle. Then, under the influence of the newly discovered mathematical science of the analysis of functions, it broadened out into the study of the simple abstract periodic functions which these ratios exemplify. Thus trigonometry became completely abstract; and in thus becoming abstract, it became useful. It illuminated the underlying analogy between sets of utterly diverse physical phenomena; and at the same time it supplied the weapons by which any one such set could have its various features analysed and related to each other[1].

Nothing is more impressive than the fact that as mathematics withdrew increasingly into the upper regions of ever greater extremes of abstract thought, it returned back to earth with a corresponding growth of importance for the analysis of concrete fact. The history of the seventeenth century science reads as though it were some vivid dream of Plato or Pythagoras. In this characteristic the seventeenth century was only the forerunner of its successors.

The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact. As the result of the prominence

[1] For a more detailed consideration of the nature and function of pure mathematics cf. my *Introduction to Mathematics*, Home University Library, Williams and Norgate, London.

of mathematicians in the seventeenth century, the eighteenth century was mathematically minded, more especially where French influence predominated. An exception must be made of the English empiricism derived from Locke. Outside France, Newton's direct influence on philosophy is best seen in Kant, and not in Hume.

In the nineteenth century, the general influence of mathematics waned. The romantic movement in literature, and the idealistic movement in philosophy were not the products of mathematical minds. Also, even in science, the growth of geology, of zoology, and of the biological sciences generally, was in each case entirely disconnected from any reference to mathematics. The chief scientific excitement of the century was the Darwinian theory of evolution. Accordingly, mathematicians were in the background so far as the general thought of that age was concerned. But this does not mean that mathematics was being neglected, or even that it was uninfluential. During the nineteenth century pure mathematics made almost as much progress as during all the preceding centuries from Pythagoras onwards. Of course progress was easier, because the technique had been perfected. But allowing for that, the change in mathematics between the years 1800 and 1900 is very remarkable. If we add in the previous hundred years, and take the two centuries preceding the present time, one is almost tempted to date the foundation of mathematics somewhere in the last quarter of the seventeenth century. The period of the discovery of the elements stretches from Pythagoras to Descartes, Newton, and Leibniz, and the developed science has been created during the last two hundred and fifty years. This is not a boast as to the superior genius of the modern world; for it is harder to discover the elements than to develop the science.

Throughout the nineteenth century, the influence of the science was its influence on dynamics and physics, and thence derivatively on engineering and chemistry. It is difficult to overrate its indirect influence on human life through the medium of these sciences. But there was no direct influence of mathematics upon the general thought of the age.

In reviewing this rapid sketch of the influence of mathematics throughout European history, we see that it had two great periods of direct influence upon general thought, both periods lasting for about two hundred years. The first period was that stretching from Pythagoras to Plato, when the possibility of the science, and its general character, first dawned upon the Grecian thinkers. The second period comprised the seventeenth and eighteenth centuries of our modern epoch. Both periods had certain common characteristics. In the earlier, as in the later period, the general categories of thought in many spheres of human interest, were in a state of disintegration. In the age of Pythagoras, the unconscious Paganism, with its traditional clothing of beautiful ritual and of magical rites, was passing into a new phase under two influences. There were waves of religious enthusiasm, seeking direct enlightenment into the secret depths of being; and at the opposite pole, there was the awakening of critical analytical thought, probing with cool dispassionateness into ultimate meanings. In both influences, so diverse in their outcome, there was one common element—an awakened curiosity, and a movement towards the reconstruction of traditional ways. The pagan mysteries may be compared to the Puritan reaction and to the Catholic reaction; critical scientific interest was alike in both epochs, though with minor differences of substantial importance.

In each age, the earlier stages were placed in periods of rising prosperity, and of new opportunities. In this respect, they differed from the period of gradual declension in the second and third centuries when Christianity was advancing to the conquest of the Roman world. It is only in a period, fortunate both in its opportunities for disengagement from the immediate pressure of circumstances, and in its eager curiosity, that the Age-Spirit can undertake any direct revision of those final abstractions which lie hidden in the more concrete concepts from which the serious thought of an age takes its start. In the rare periods when this task can be undertaken, mathematics becomes relevant to philosophy. For mathematics is the science of the most complete abstractions to which the human mind can attain.

The parallel between the two epochs must not be pressed too far. The modern world is larger and more complex than the ancient civilisation round the shores of the Mediterranean, or even than that of the Europe which sent Columbus and the Pilgrim Fathers across the ocean. We cannot now explain our age by some simple formula which becomes dominant and will then be laid to rest for a thousand years. Thus the temporary submergence of the mathematical mentality from the time of Rousseau onwards appears already to be at an end. We are entering upon an age of reconstruction, in religion, in science, and in political thought. Such ages, if they are to avoid mere ignorant oscillation between extremes, must seek truth in its ultimate depths. There can be no vision of this depth of truth apart from a philosophy which takes full account of those ultimate abstractions, whose inter-connections it is the business of mathematics to explore.

In order to explain exactly how mathematics is gaining in general importance at the present time, let us start from a particular scientific perplexity and consider the notions to which we are naturally led by some attempt to un-

ravel its difficulties. At present physics is troubled by the quantum theory. I need not now explain[1] what this theory is, to those who are not already familiar with it. But the point is that one of the most hopeful lines of explanation is to assume that an electron does not continuously traverse its path in space. The alternative notion as to its mode of existence is that it appears at a series of discrete positions in space which it occupies for successive durations of time. It is as though an automobile moving at the average rate of thirty miles an hour along a road, did not traverse the road continuously; but appeared successively at the successive milestones, remaining for two minutes at each milestone.

In the first place there is required the purely technical use of mathematics to determine whether this conception does in fact explain the many perplexing characteristics of the quantum theory. If the notion survives this test, undoubtedly physics will adopt it. So far the question is purely one for mathematics and physical science to settle between them, on the basis of mathematical calculations and physical observations.

But now a problem is handed over to the philosophers. This discontinuous existence in space, thus assigned to electrons, is very unlike the continuous existence of material entities which we habitually assume as obvious. The electron seems to be borrowing the character which some people have assigned to the Mahatmas of Tibet. These electrons, with the correlative protons, are now conceived as being the fundamental entities out of which the material bodies of ordinary experience are composed. Accordingly if this explanation is allowed, we have to revise all our notions of the ultimate character of material existence. For when we

[1] Cf. Chapter VIII.

penetrate to these final entities, this startling discontinuity of spatial existence discloses itself.

There is no difficulty in explaining the paradox, if we consent to apply to the apparently steady undifferentiated endurance of matter the same principles as those now accepted for sound and light. A steadily sounding note is explained as the outcome of vibrations in the air: a steady colour is explained as the outcome of vibrations in ether. If we explain the steady endurance of matter on the same principle, we shall conceive each primordial element as a vibratory ebb and flow of an underlying energy, or activity. Suppose we keep to the physical idea of energy: then each primordial element will be an organised system of vibratory streaming of energy. Accordingly there will be a definite period associated with each element; and within that period the stream-system will sway from one stationary maximum to another stationary maximum,—or, taking a metaphor from the ocean tides, the system will sway from one high tide to another high tide. This system, forming the primordial element, is nothing at any instant. It requires its whole period in which to manifest itself. In an analogous way, a note of music is nothing at an instant, but it also requires its whole period in which to manifest itself.

Accordingly, in asking where the primordial element is, we must settle on its average position at the centre of each period. If we divide time into smaller elements, the vibratory system as one electronic entity has no existence. The path in space of such a vibratory entity—where the entity is *constituted by* the vibrations—must be represented by a series of detached positions in space, analogously to the automobile which is found at successive milestones and at nowhere between.

We first must ask whether there is any evidence to

associate the quantum theory with vibration. This question is immediately answered in the affirmative. The whole theory centres round the radiant energy from an atom, and is intimately associated with the periods of the radiant wave-systems. It seems, therefore, that the hypothesis of essentially vibratory existence is the most hopeful way of explaining the paradox of the discontinuous orbit.

In the second place, a new problem is now placed before philosophers and physicists, if we entertain the hypothesis that the ultimate elements of matter are in their essence vibratory. By this I mean that apart from being a periodic system, such an element would have no existence. With this hypothesis we have to ask, what are the ingredients which form the vibratory organism. We have already got rid of the matter with its appearance of undifferentiated endurance. Apart from some metaphysical compulsion, there is no reason to provide another more subtle stuff to take the place of the matter which has just been explained away. The field is now open for the introduction of some new doctrine of organism which may take the place of the materialism with which, since the seventeenth century, science has saddled philosophy. It must be remembered that the physicists' energy is obviously an abstraction. The concrete fact, which is the organism, must be a complete expression of the character of a real occurrence. Such a displacement of scientific materialism, if it ever takes place, cannot fail to have important consequences in every field of thought.

Finally, our last reflection must be, that we have in the end come back to a version of the doctrine of old Pythagoras, from whom mathematics, and mathematical physics, took their rise. He discovered the importance of dealing with abstractions; and in particular directed attention to number as characterising the periodicities of notes of music. The importance of the abstract idea of periodicity was thus present at the very beginning both of mathematics and of European philosophy.

In the seventeenth century, the birth of modern science required a new mathematics, more fully equipped for the purpose of analysing the characteristics of vibratory existence. And now in the twentieth century we find physicists largely engaged in analysing the periodicities of atoms. Truly, Pythagoras in founding European philosophy and European mathematics, endowed them with the luckiest of lucky guesses—or, was it a flash of divine genius, penetrating to the inmost nature of things?

CHAPTER III

THE CENTURY OF GENIUS

THE previous chapters were devoted to the antecedent conditions which prepared the soil for the scientific outburst of the seventeenth century. They traced the various elements of thought and instinctive belief, from their first efflorescence in the classical civilisation of the ancient world, through the transformations which they underwent in the Middle Ages, up to the historical revolt of the sixteenth century. Three main factors arrested attention,—the rise of mathematics, the instinctive belief in a detailed order of nature, and the unbridled rationalism of the thought of the later Middle Ages. By this rationalism I mean the belief that the avenue to truth was predominantly through a metaphysical analysis of the nature of things, which would thereby determine how things acted and functioned. The historical revolt was the definite abandonment of this method in favour of the study of the empirical fact of antecedents and consequences. In religion, it meant the appeal to the origins of Christianity; and in science it meant the appeal to experiment and the inductive method of reasoning.

A brief, and sufficiently accurate, description of the intellectual life of the European races during the succeeding two centuries and a quarter up to our own times is that they have been living upon the accumulated capital of ideas provided for them by the genius of the seventeenth century. The men of this epoch inherited a ferment of ideas attendant upon the historical revolt of the sixteenth century, and they bequeathed formed systems of thought touching every aspect